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Probability current versus charge current of a relativistic particle†

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Abstract. Newton and Wigner found the probability density of detecting a relativistic particle in some point in space. This probability density must, however, be the zero component of the same vector field. This (unique) vector field—the probability current—is constructed and discussed.

1. Introduction

In this paper we shall be concerned (for the sake of simplicity) with the description of a free, relativistic spinless particle. Such a particle is described in quantum field theory by a vector belonging to the one-particle sector $\mathscr{H}^{(1)}$ of the Fock space [3]. The subspace $\mathscr{H}^{(1)}$ is spanned by the basis elements

$$|\boldsymbol{k}\rangle = a_{\boldsymbol{k}}^{+}|0\rangle \tag{1}$$

in the momentum representation, where $[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^{+}] = \delta(\boldsymbol{k}-\boldsymbol{k}')$. The state vector at some moment t is a vector $|\psi_t\rangle$:

$$|\psi_t\rangle = \int \mathrm{d}\boldsymbol{k}\, \psi(\boldsymbol{k}, t)|\boldsymbol{k}\rangle. \tag{2}$$

The probability density of detecting the particle (at time t) with a momentum $\boldsymbol{k}$ is

$$\rho(\boldsymbol{k}, t) = |\psi(\boldsymbol{k}, t)|^2. \tag{3}$$

The function $\psi(\boldsymbol{k}, t)$ will be called the Newton-Wigner-Foldy [1, 2] (NWF) wavefunction; it is the relativistic analogue of the Schrödinger wavefunction in the momentum representation. The function $\psi(\boldsymbol{k}, t)$ is normalised by

$$\int \mathrm{d}\boldsymbol{k} |\psi(\boldsymbol{k}, t)|^2 = 1. \tag{4}$$

The evolution in time is governed by the Hamiltonian

$$H = \int \mathrm{d}\boldsymbol{k}\, \omega_{\boldsymbol{k}} a_{\boldsymbol{k}}^{+} a_{\boldsymbol{k}} \tag{5}$$

where

$$\omega_{\boldsymbol{k}} \equiv (\boldsymbol{k}^2 + m^2)^{1/2}. \tag{6}$$

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Therefore in the one-particle sector $\mathscr{H}^{(1)}$ the basis vector $|\boldsymbol{k}\rangle$ evolves during the time t to the vector

$$\exp(\mathrm{i}Ht)|\boldsymbol{k}\rangle = \exp(\mathrm{i}\omega_k t)|\boldsymbol{k}\rangle \tag{7}$$

and then the vector described at the time $t=0$ by the NWF wavefunction $\psi(\boldsymbol{k}, t=0)$ changes during the time t to

$$\psi(\boldsymbol{k}, t) = \exp(-\mathrm{i}\omega_k t)\psi(\boldsymbol{k}, 0). \tag{8}$$

Therefore the equation of motion is

$$\mathrm{i}(\partial/\partial t)\psi(\boldsymbol{k}, t) = \omega_k\psi(\boldsymbol{k}, t) = (\boldsymbol{k}^2+m^2)^{1/2}\psi(\boldsymbol{k}, t). \tag{9}$$

The equations of that type were derived (in a different way) by Foldy [2] and will be called Foldy equations. They are the relativistic analogues of the Schrödinger equation. Now let us go to the coordinate representation. In place of the basic elements $|\boldsymbol{k}\rangle$ there are the basic elements

$$|\boldsymbol{x}\rangle \equiv \int \mathrm{d}\boldsymbol{k}\,\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})|\boldsymbol{k}\rangle \tag{10}$$

and in place of the NWF wavefunction $\psi(\boldsymbol{k}, t)$ there is $\psi(\boldsymbol{x}, t)$

$$\psi(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{k}\,\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\psi(\boldsymbol{k}, t) \tag{11}$$

so that

$$|\psi_t\rangle = \int \mathrm{d}\boldsymbol{k}\,\psi(\boldsymbol{k}, t)|\boldsymbol{k}\rangle = \int \mathrm{d}\boldsymbol{x}\,\psi(\boldsymbol{x}, t)|\boldsymbol{x}\rangle. \tag{12}$$

The states $|\boldsymbol{x}\rangle$ are the Newton–Wigner 'localised' states [1]. The probability density to find a particle at the point $\boldsymbol{x}$ is

$$\rho(\boldsymbol{x}, t) = |\psi(\boldsymbol{x}, t)|^2. \tag{13}$$

The Foldy equation (9) in x-representation is non-local, i.e.,

$$\mathrm{i}(\partial/\partial t)\psi(\boldsymbol{x}, t) = (-\nabla^2+m^2)^{1/2}\psi(x, t) = \int \mathrm{d}\boldsymbol{y}\,K(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y}, t) \tag{14}$$

where

$$K(\boldsymbol{x}) \equiv \int \mathrm{d}\boldsymbol{k}\,\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})(k^2+m^2)^{1/2}.$$

The probability density $\rho(\boldsymbol{x}, t)$ is the zero component of a vector field J_μ [4]. Our goal is to find three other components of this vector field.

2. The derivation of the probability 4-current

We know $\rho(\boldsymbol{x}, t) = J_0(\boldsymbol{x}, t)$ in *any* frame of reference. $J_\mu(x)$ transforms under any Lorentz transformation Λ as a vector field

$$J'_\mu(\Lambda x) = \Lambda_\mu{}^v J_v(x). \tag{15}$$

The NWF wavefunction $\psi(\boldsymbol{x}, t)$ does not have a simple Lorentz transformation, but the function

$$\phi(x) = \omega^{-1/2}\psi(x) = \int \mathrm{d}\boldsymbol{y}\, K^{-1/2}(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y}, t), \tag{16}$$

where $K^{\alpha} \equiv \int \mathrm{d}\boldsymbol{x}\, \omega^{\alpha} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$, is a scalar [4], i.e.,

$$\phi'(\Lambda x) = \phi(x). \tag{17}$$

The expression of $J_0(x)$ in terms of $\phi(x)$ is

$$J_0(x) = |\omega^{1/2}\phi(x)|^2 = \left| \int \mathrm{d}\boldsymbol{y}\, K^{1/2}(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y}, t)\right|^2. \tag{18}$$

The equation (15) for the infinitesimal pure boast

$$\Lambda_{00} = 1, \qquad \Lambda_{0i} = -v_i, \qquad \Lambda_{i0} = -v_i, \qquad \Lambda_{ij} = \delta_{ij}, \tag{19}$$

and for $\mu = 0$ in (15) takes the form

$$J_0'(\boldsymbol{x}-\boldsymbol{v}t, t-\boldsymbol{v}\boldsymbol{x}) = J_0(\boldsymbol{x}, t) - \boldsymbol{v}\boldsymbol{J}(\boldsymbol{x}, t). \tag{20}$$

We expand the left-hand side of (20) in the small parameter $\boldsymbol{v}$:

$$\begin{aligned}
&J_0'(\boldsymbol{x}-\boldsymbol{v}t, t-\boldsymbol{v}\boldsymbol{x})\\
&\quad = |\psi'(\boldsymbol{x}', t')|^2 = \left|\int K^{1/2}(\boldsymbol{x}'-\boldsymbol{y}')\phi'(\boldsymbol{y}', t')\,\mathrm{d}\boldsymbol{y}'\right|^2\\
&\quad = \left|\int \mathrm{d}\boldsymbol{y}'\, K^{1/2}(\boldsymbol{x}'-\boldsymbol{y}')\phi(\boldsymbol{y}'+\boldsymbol{v}t, t'+\boldsymbol{v}\boldsymbol{y}')\right|^2\\
&\quad \approx \left|\int \mathrm{d}\boldsymbol{y}\, K^{1/2}(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y}, t'+\boldsymbol{v}\boldsymbol{y})\right|^2\\
&\quad = \left|\int \mathrm{d}\boldsymbol{y}\, K^{1/2}(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y}, t-\boldsymbol{v}(\boldsymbol{x}-\boldsymbol{y}))\right|^2\\
&\quad \approx \left|\int \mathrm{d}\boldsymbol{y}\, K^{1/2}(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y}, t) - \boldsymbol{v}\int \mathrm{d}\boldsymbol{y}(\boldsymbol{x}-\boldsymbol{y})K^{1/2}(\boldsymbol{x}-\boldsymbol{y})\dot{\phi}(\boldsymbol{y}, t)\right|^2\\
&\quad = |\psi(\boldsymbol{x}, t) - \mathrm{i}\boldsymbol{v}((\partial/\partial\boldsymbol{k})\omega^{1/2})\dot{\phi}(\boldsymbol{x}, t)|^2\\
&\quad = |\psi(\boldsymbol{x}, t) + \tfrac{1}{2}\boldsymbol{v}(\boldsymbol{k}/\omega^{1/2})\phi(\boldsymbol{x}, t)|^2\\
&\quad = |\psi(\boldsymbol{x}, t) + \tfrac{1}{2}\boldsymbol{v}(\boldsymbol{k}/\omega)\psi(\boldsymbol{x}, t)|^2\\
&\quad = |\psi(\boldsymbol{x}, t)|^2 + \tfrac{1}{2}\boldsymbol{v}(\psi^*(\overrightarrow{\boldsymbol{k}/\omega})\psi + \psi^*(\overleftarrow{\boldsymbol{k}/\omega})\psi).
\end{aligned} \tag{21}$$

From (20) and (21) it follows that

$$\boldsymbol{J} = \tfrac{1}{2}[\psi^*(\overrightarrow{\boldsymbol{k}/\omega})\psi + \psi^*(\overleftarrow{\boldsymbol{k}/\omega})\psi] = \tfrac{1}{2}[\phi^*(\overleftarrow{\omega^{1/2}})(\overrightarrow{\boldsymbol{k}/\omega})\phi + \phi^*(\overleftarrow{\boldsymbol{k}/\omega})(\overrightarrow{\omega^{1/2}})\phi]. \tag{22}$$

Combined with (18) we have

$$J_\mu = \tfrac{1}{2}[\psi^*(\overrightarrow{k_\mu/\omega})\psi + \psi^*(\overleftarrow{k_\mu/\omega})\psi]. \tag{23}$$

3. Comparison with the charge current

In the one-particle state $|\psi\rangle$ the charge current operator

$$\hat{j}_\mu(x) = -\tfrac{1}{2}\mathrm{i} :\hat{\varphi}^{+}\overleftrightarrow{\partial}_\mu\hat{\varphi}: \tag{24}$$

where $\hat{\varphi}$ is a scalar field operator, and we have indicated normal ordering, has the expectation value

$$\begin{aligned} j_0(x) &= -\tfrac{1}{2}\mathrm{i}\phi^*(\overleftrightarrow{\partial/\partial t})\phi = \tfrac{1}{2}[\psi^*(\overleftarrow{\omega^{-1/2}})(\overrightarrow{\omega^{1/2}})\psi + \psi^*(\overleftarrow{\omega^{1/2}})(\overrightarrow{\omega^{-1/2}})\psi \\ j(x) &= \tfrac{1}{2}\phi^*(\overleftrightarrow{k})\phi = \tfrac{1}{2}[\psi^*(\overleftarrow{\omega^{-1/2}})(\overrightarrow{k/\omega^{1/2}})\psi + \psi^*(\overleftarrow{k/\omega^{1/2}})(\overrightarrow{\omega^{-1/2}})\psi. \end{aligned} \tag{25}$$

The quantity is called the charge current (clearly not proportional to the probability current). The function j_0 is normalised to 1 when ψ is normalised

$$\begin{aligned} \int \mathrm{d}\boldsymbol{x}\, j_0(\boldsymbol{x}, t) &= \tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}[\psi^*(\overleftarrow{\omega^{-1/2}})(\overrightarrow{\omega^{1/2}})\psi + \psi^*(\overleftarrow{\omega^{1/2}})(\overrightarrow{\omega^{-1/2}})\psi] \\ &= \tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}[\psi^*(\boldsymbol{k})\omega^{-1/2}\omega^{1/2}\psi(\boldsymbol{k}) + \psi^*(\boldsymbol{k})\omega^{1/2}\omega^{-1/2}\psi(\boldsymbol{k})] = \int \mathrm{d}\boldsymbol{k}|\psi(\boldsymbol{k})|^2 = 1. \end{aligned} \tag{26}$$

The only states for which the probability current is equal to the charge current are plane waves. For other one-particle states, the two types of currents are different. An interesting feature of the charge density of one positively charged particle is that in spite of the normalisation (26) it may be negative in some regions. On the other hand, the probability density $J_0(\boldsymbol{x}, t)$ is obviously positive everywhere. Another feature is that j_μ obeys the continuity equation

$$\partial_\mu j^\mu = 0. \tag{27}$$

The probability current has the property

$$\partial_\mu J^\mu = \tfrac{1}{2}\mathrm{i}[m^2(\psi^*(\overleftrightarrow{1/\omega})\psi + \psi^*(\overleftarrow{k_\mu})(\overrightarrow{k_\mu/\omega})\psi + \psi^*(\overleftarrow{k_\mu/\omega})(\overrightarrow{k_\mu})\psi)] \neq 0. \tag{28}$$

The conservation of the number of particles (equal to 1 in our case), of course, is satisfied globally by $J_0(x)$, but this is not enough to imply the local continuity equation. It is always possible to find many 3-vectors $\boldsymbol{J}(\boldsymbol{x}, t)$ that satisfy

$$(\partial/\partial t)J_0 = \boldsymbol{\nabla}\boldsymbol{J} \tag{29}$$

but these $\boldsymbol{J}$ cannot in general be a part of a 4-vector, and therefore have no covariant meaning.

4. Interpretation of the probability current

As we have seen, probability current is not divergenceless. Let us find the restrictions on the divergence following from the conservation of number of particles.

Since J^μ is formed of wavefunctions which are square integrable in space,

$$\int \mathrm{d}^3x\, \Omega(x, t) \equiv \int \mathrm{d}^3x\, \partial_\mu J^\mu(x, t) = (\partial/\partial t)\left[\int \mathrm{d}^3x\, J^0(x, t)\right]. \tag{30}$$

The integral on J^0 is, however, the total probability of finding the particle, and hence is normalised to 1 at any t. This implies that

$$\int d^3x\, \Omega(x, t) = 0. \tag{31}$$

The invariance implies that the integral of $\Omega(x, t)$ over *any* space like plane σ should also vanish. As a consequence of (31) and of the fact that $J^0 = 0$ at $x \to \infty$ we can state that the four-dimensional integral of Ω over the four-dimensional volume between two space-like surfaces σ_1 and σ_2 is zero:

$$\int_{\sigma_1}^{\sigma_2} d^4x\, \Omega = \int_{\sigma_1}^{\sigma_2} d^4x\, \partial_\mu J^\mu = \int_{\sigma_2} \frac{\partial}{\partial t} J^0 - \int_{\sigma_1} \frac{\partial}{\partial t} J^0 = 0. \tag{32}$$

In particular the integral of Ω over whole spacetime vanishes.

As an illustration of the construction of such a function Ω, consider the evolution of the 'Lorentzian' wavepacket (see figure 1)

$$\psi_0(x) = (2/\pi)^{1/2} \gamma_0^{3/2} / (\gamma_0^2 + x^2) \tag{33}$$

in one space dimension and in the ultrarelativistic regime

$$m \ll 1/\gamma_0. \tag{34}$$

The Fourier transform of $\psi_0(x)$ is

$$\psi_0(k) = \gamma_0^{1/2} \exp(-\gamma |k|). \tag{35}$$

The indeterminacy of the Lorentzian wavepacket is

$$\Delta x \Delta p = 2^{-1/2} \hbar. \tag{36}$$

Now we can calculate (neglecting mass m) the wavefunction at any time according to (8) and (11)

$$\psi_t(x) = \int_{-\infty}^{\infty} dk \exp(-ixk) \psi_0(k) \exp(-it|k|) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\gamma_0^{1/2} \gamma}{\gamma^2 + x^2} \tag{37}$$

where

$$\gamma \equiv \gamma_0 - it. \tag{38}$$

The probability density J^0 and probability current J^1 are

$$J^0 = \frac{2}{\pi} \frac{\gamma_0 \gamma \gamma^*}{(\gamma^2 + x^2)(\gamma^{*2} + x^2)} \tag{39}$$

$$J' = \frac{2}{\pi} \frac{tx(\gamma\gamma^*)^{1/2}}{(\gamma^2 + x^2)(\gamma^{*2} + x^2)}. \tag{40}$$

The wavepacket spreads inside the light cone as shown in figure 1. The divergence is

$$\Omega = \frac{2}{\pi} \frac{t[-x^4 + 2x^2 t^2 + 6x^2 - (1 + t^2)^2]}{(\gamma^2 + x^2)^2 (\gamma^{*2} + x^2)^2}. \tag{41}$$

The regions of $\Omega > 1$ and $\Omega < -1$ on the spacetime map are shown on figure 2.

From the point of view of spacetime, the divergence of a current 4-vector corresponds to the source for which the current is the flux J^μ. As in the Poisson equation, $\nabla E = \rho$, one could consider the flux to be a static field strength. However, because of

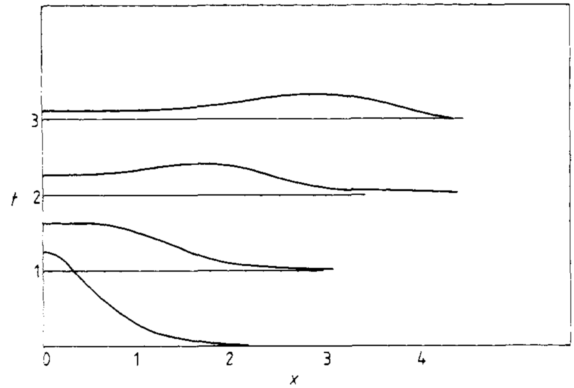

Figure 1. The change in probability density of relativistic particle with time.

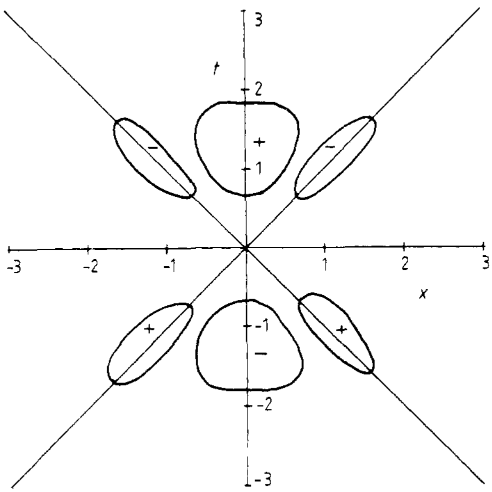

Figure 2. The spacetime localisation of Ω.

the physical nature of the current, it is natural to consider it as a flow of events in spacetime, with source corresponding to the rate of increase in the density of events locally.

5. The non-relativistic limit

In the non-relativistic limit the probability density and the charge density coincide. The Schrödinger wavefunction $\psi_{nr}(x, t)$ is NWF wavefunction, but for convenience an immeasurable phase is introduced

$$\psi_{nr}(\boldsymbol{x}, t) \equiv \lim_{c\to\infty} \exp(imc^2 t)\psi(\boldsymbol{x}, t) \tag{42}$$

to simplify the equations. The equation (14) in the non-relativistic limit becomes the

Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi_{\mathrm{nr}}=[(-\nabla^2c^2+m^2c^4)^{1/2}-mc^2]\psi_{\mathrm{nr}}=-(\nabla^2/2m)\psi_{\mathrm{nr}}+\mathrm{O}(1/c^2). \tag{43}$$

The probability density is

$$J_0=|\psi|^2=(\psi_{\mathrm{nr}})^2 \tag{44}$$

and the charge density is

$$\begin{aligned} j_0&=\tfrac{1}{2}[\psi^*(\overleftarrow{\omega^{-1/2}})(\overrightarrow{\omega^{1/2}})\psi+\psi^*(\overleftarrow{\omega^{1/2}})(\overrightarrow{\omega^{-1/2}})\psi]\\ &=|\psi|^2+\mathrm{O}(1/c^2). \end{aligned} \tag{45}$$

Therefore,

$$j_0=J_0+\mathrm{O}(1/c^2). \tag{46}$$

Let us now study $\boldsymbol{j}$ and $\boldsymbol{J}$.

The non-relativistic limit of $\boldsymbol{j}$ is the well known quantity called 'probability current' in the textbooks

$$\boldsymbol{j}=\tfrac{1}{2}c[\psi^*(\overleftarrow{\omega^{-1/2}})(\overrightarrow{\boldsymbol{k}/\omega^{1/2}})\psi+\psi^*(\overleftarrow{\boldsymbol{k}/\omega^{1/2}})(\overrightarrow{\omega^{-1/2}})\psi]=-(\mathrm{i}/2m)\psi^*\overleftrightarrow{\nabla}\psi+\mathrm{O}(1/c^2). \tag{47}$$

The continuity equation for j_μ is preserved in the non-relativistic limit.

On the other hand, $\boldsymbol{J}(x)$ is

$$\boldsymbol{J}=\tfrac{1}{2}c[\psi^*(\overrightarrow{\boldsymbol{k}/\omega})\psi+\psi^*(\overleftarrow{\boldsymbol{k}/\omega})\psi]=-(\mathrm{i}/2m)\psi^*\overleftrightarrow{\nabla}\psi+\mathrm{O}(1/c^2) \tag{48}$$

and is therefore identical to $\boldsymbol{j}(x)$. In this limit, then, the probability current becomes divergenceless as well, as is clear from (27). The non-vanishing of the divergence of the relativistic probability current is, therefore, a purely relativistic effect.

6. Conclusion

The probability density for the detection of particles, represented in the one-particle sector of quantum field theory by Newton–Wigner–Foldy wavefunctions, can be completed covariantly to a 4-vector. This probability current does not coincide with the (not everywhere positive) charge current. It is, in fact, not divergenceless. In the non-relativistic limit, the two currents coincide, and the probability current becomes divergenceless. The non-vanishing divergence of the relativistic probability current is therefore a relativistic effect.

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